Supplementary Material for : Scale Dependent Distribution of Kinetic Energy from Surface Drifters in the Gulf of Mexico

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1 Relationship between the second order structure function and the energy density

In this appendix we discuss how the cross over length scale of two components of the kinetic energy spectrum, say the divergent and rotational components, are related to the cross over length scale of the corresponding second order velocity structure functions, where the cross over length scale is defined as the length scale where two components intersect and exchange dominance. As the energy spectrum represents the energy density as a function of length scales, it is beneficial to know the cross over length scale for the divergent and rotational components of the energy spectrum. We also provide a complementary discussion in terms of signature functions, which are the real space alternative to the energy spectra.

The second order velocity structure function can be related to the 2D isotropic energy spectrum as

$$D2_{ii}(r) = 4\int_0^\infty E_{ii}(k)(1 - J_0(kr))dk \qquad (1)$$

$$E_{ii}(k) = \int_0^\infty \frac{D2_{ii}(\infty) - D2_{ii}(r)}{4} J_0(kr)krdr, \qquad (2)$$

and to the 1D isotropic energy spectrum as

$$D2_{ii}(r) = 4\int_0^\infty E_{ii}(k)(1 - cos(kr))dk \qquad (3)$$

$$E_{ii}(k) = \frac{1}{2\pi}\int_0^\infty (D2_{ii}(\infty) - D2_{ii}(r))cos(kr)dr. \tag{4}$$

Subscript ii does not imply summation here. These relationships have been derived previously in Babiano et al 1985, Bennett 1985 and LaCasce 2016. LaCasce 2016 also showed that the transform of the energy spectrum to the structure function is straight forward and well behaved for both idealized and observational data. However, the inverse transform is sensitive to the details at large separations and produces very noisy results. We attempted to transform the GLAD structure functions to the corresponding energy spectrum, but were unsuccessful in producing any meaningful results.

Due to the integral nature of the relationship between the structure function and the energy spectrum, the cross over length scales observed using the second order velocity structure function and the energy spectrum would be different. However, as shown below for an idealized scenario, the difference might be minor for some cases.

We consider an idealized power law spectrum

$$E_\alpha(k) = C_\alpha(k/k_o)^{-\alpha}, \tag{5}$$

with a finite range $k_o < k < k_1$, where k_o and k_1 are the lower and higher wave number cutoffs.

This energy spectrum can be transformed to the corresponding structure functions using equation 1. Here, we use an approximation to equation 1,

$$D2(r) \approx 4(\frac{r^2}{4}\int_0^{2/r} k^2E(k)dk + \int_{2/r}^\infty E(k)dk), \tag{6}$$

however the results are not sensitive to this choice.

This approximate can be easily integrated with the above choice of $E_\alpha(k)$ and results in

$$D2_\alpha(r) = \frac{C_\alpha k_o^\alpha}{(3-\alpha)} r^2 k^{3-\alpha}|_{k_o}^{2/r} + 4\frac{C_\alpha k_o^\alpha}{1-\alpha}k^{1-\alpha}|_{2/r}^{k_1}. \tag{7}$$

Two energy spectra (E_α and E_β) with different slopes intersect at the wavenumber $k_i = k_o(C_\alpha/C_\beta)^{1/(\alpha-\beta)}$, and a corresponding length scale that can be defined as π/k_i. The intersection length scale (r_i) for the corresponding second order velocity structure functions ($D2_\alpha(r)$ and $D2_\beta(r)$) can found numerically by finding the roots to $D2_\alpha(r) - D2_\beta(r) = 0$.

An idealized case was considered with a fixed α of 5/3, representing the divergent component, and β that is varied from 5/3 to 3, representing a range of slopes for the rotational component. The wavenumber where the two spectra intersect was also varied from k_o to k_1. The ratio of the two cross over length scales ($\pi/(k_i r_i)$) is generally order 1, as shown in Figure S7. Thus, for idealized scenarios where the structure functions or energy spectrum follow the same power law behavior over a range of scales, the cross over length scale determined from either measure are similar. This might be applicable to the GLAD results, where the rotational and divergent component seem to have a constant power law behavior over 100m - 10km range of scales.

Davidson and Pearson 2005 define a real space alternative to the energy spectrum, the "signature function". For 2D (not necessarily non-divergent) isotropic flows the signature function is defined as

$$V(r) = \frac{-r^2}{4}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial D2(r)}{\partial r}. \tag{8}$$

Calculating signature functions for a given second order velocity structure function is straightforward, but the estimate can be very noisy for observations due to the additional derivatives in the definition of the signature function. In Figure S8, we present a smoothed version of second order velocity structure functions from the GLAD drifter data and the corresponding signature functions ($rV(r)$ is the measure of energy density). The smoothing shifts the cross over length scale for the structure functions to a slightly smaller length scale ($\sim$ 2 km), compared to the original structure function estimates (Figure 2, main text). The estimated signature function is noisy, but indicates a cross over length scale that is similar to the one seen using the structure functions.

Thus, signature functions support the earlier result using idealized energy spectra, the second order velocity structure functions potentially provide a length scale at which the energy density of divergent and rotational flows exchange dominance.

Finally, even though these results indicate that the cross over length scales are comparable in this study, this might not always be the case (Callies et al 2016). If possible, investigators should calculate both the energy spectrum and structure functions before drawing conclusions.

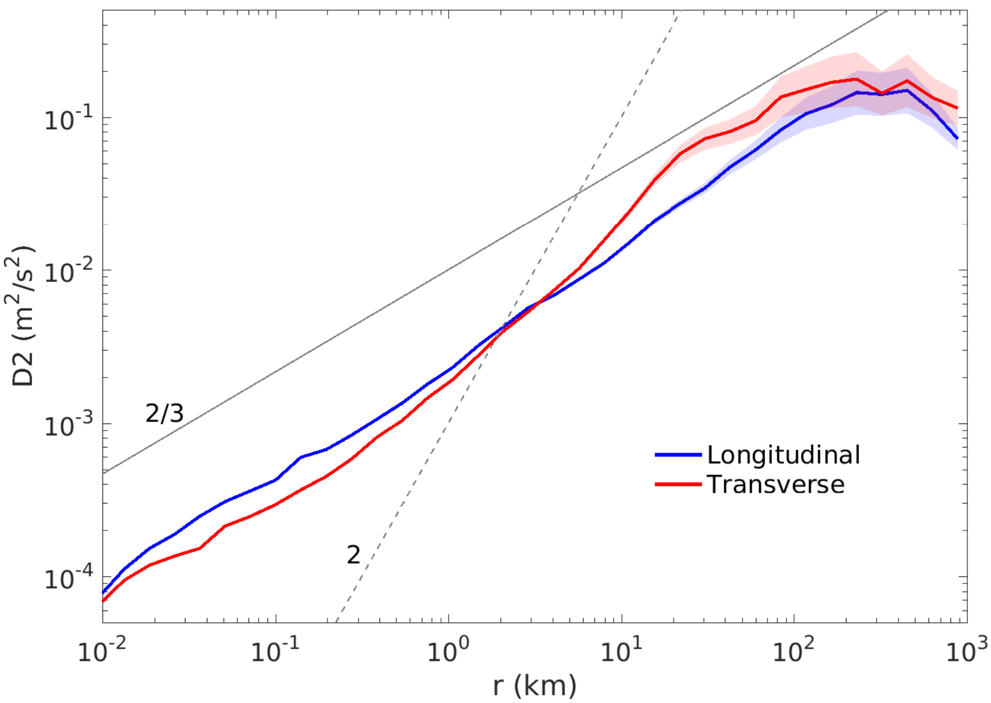

Figure S1: The longitudinal and transverse second order velocity structure functions with errorbars estimated as the 95th percentile confidence intervals using bootstrapping.

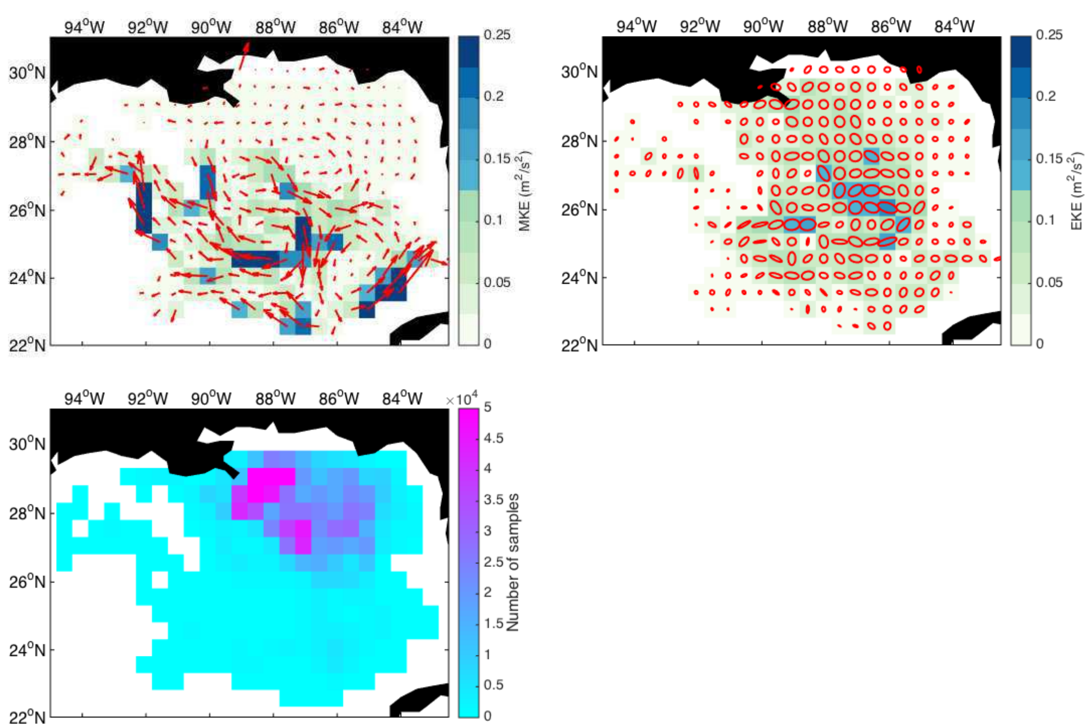

Figure S2: **Binned statistics in 0.5X0.5 degree bins calculated using the GLAD data in the Gulf of Mexico** (a) Mean kinetic energy and velocity vectors, (b) eddy kinetic energy and variance ellipses, and (c) number of drifter samples per bin.

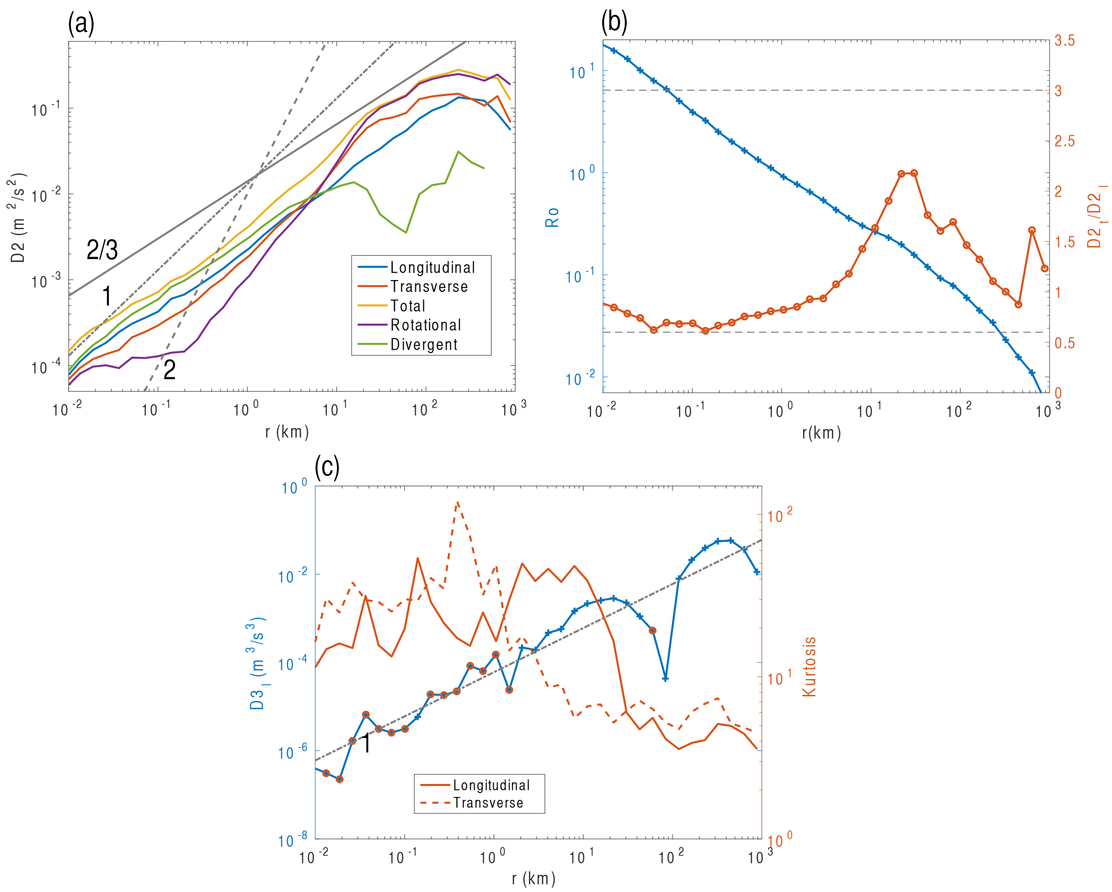

Figure S3: **Results for drifters trajectories situated to the north and west of 87W and 27N.** (a) Different components of the second order velocity structure functions. Three power law relationships are plotted as gray lines with slopes marked on the top. (b) Left axis, blue, shows the Rossby number defined as $\sqrt{D2}/(fr)$, where f is the Coriolis parameter and r is the separation distance. Right axis, orange, shows the ratio of the transverse to longitudinal second order velocity structure function ($D2_t/D2_l$). (c) Absolute value of the longitudinal component of the third order velocity structure function ($D3_l$) on the left axis (blue axis and line). Orange circles and blue pluses represent negative and positive values, respectively. A linear power law relationship is shown as dashed gray line. Kurtosis ($\frac{<\delta u^4>}{<\delta u^2>^2}$), both longitudinal and transverse components, is shown in orange on the right axis.

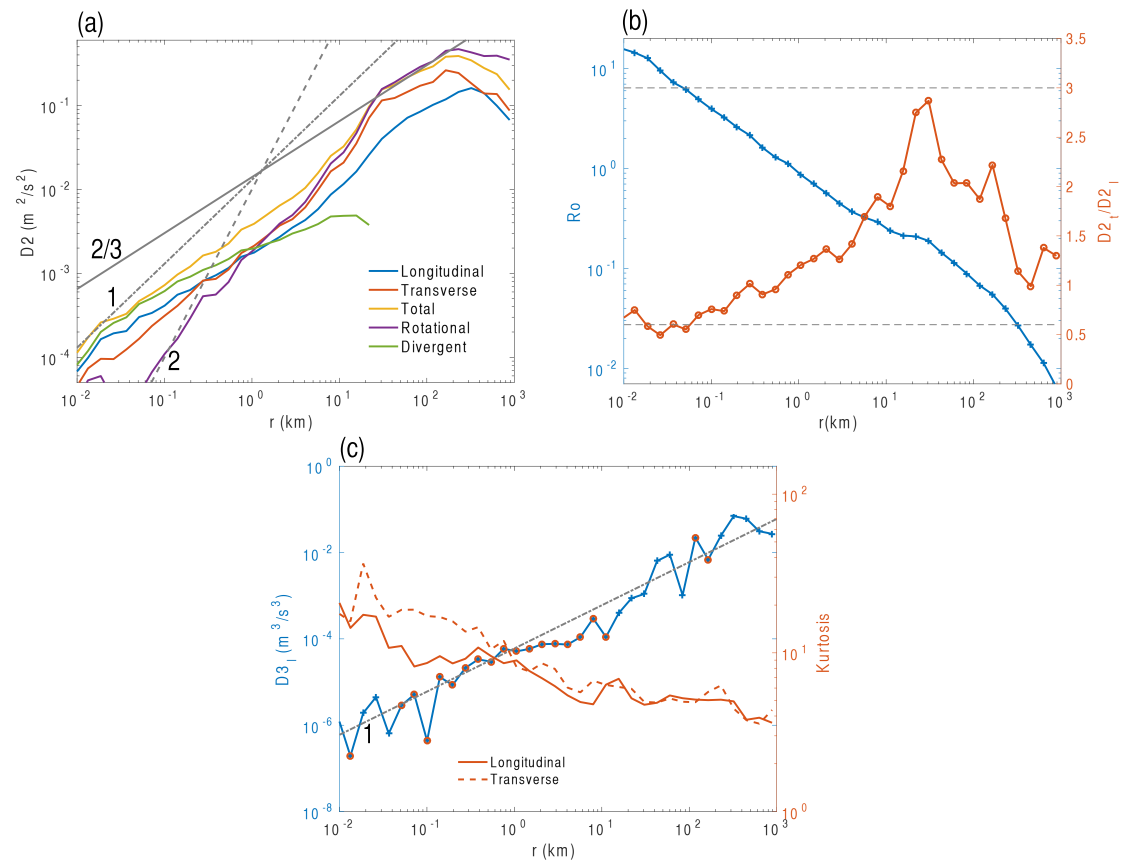

Figure S4: **Results for drifters trajectories situated to the south and east of 87W and 27N.** (a), (b) and (c) are the same as for Figure S3.

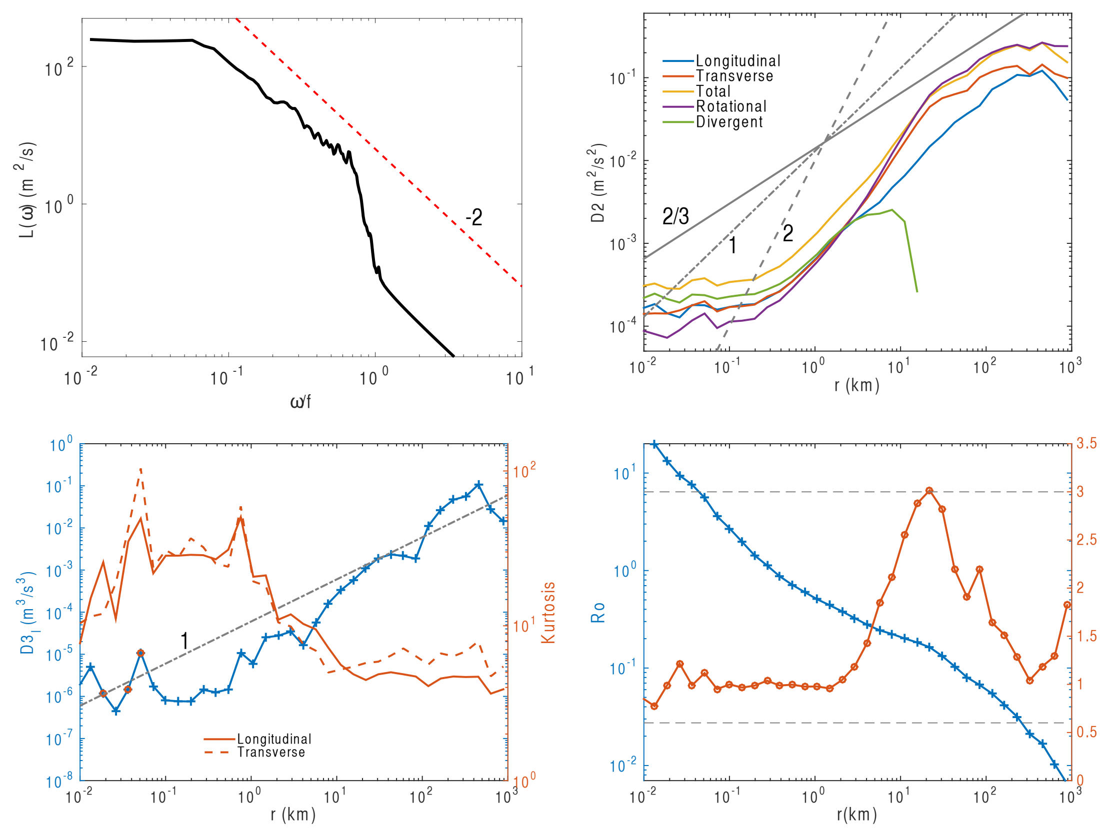

Figure S5: **Lowpass filtered results** (a) Lagrangian frequency spectrum after filtering. (b) Different components of the second order velocity structure functions. Three power law relationships are plotted as gray lines with slopes marked on the top. (c) Left axis, blue, shows the Rossby number defined as $\sqrt{D2}/(fr)$, where f is the Coriolis parameter and r is the separation distance. Right axis, orange, shows the ratio of the transverse to longitudinal second order velocity structure function ($D2_t/D2_l$). (d) Absolute value of the longitudinal component of the third order velocity structure function ($D3_l$) on the left axis (blue axis and line). Orange circles and blue pluses represent negative and positive values, respectively. A linear power law relationship is shown as dashed gray line. Kurtosis ($\frac{<\delta u^4>}{<\delta u^2>^2}$), both longitudinal and transverse components, is shown in orange on the right axis.

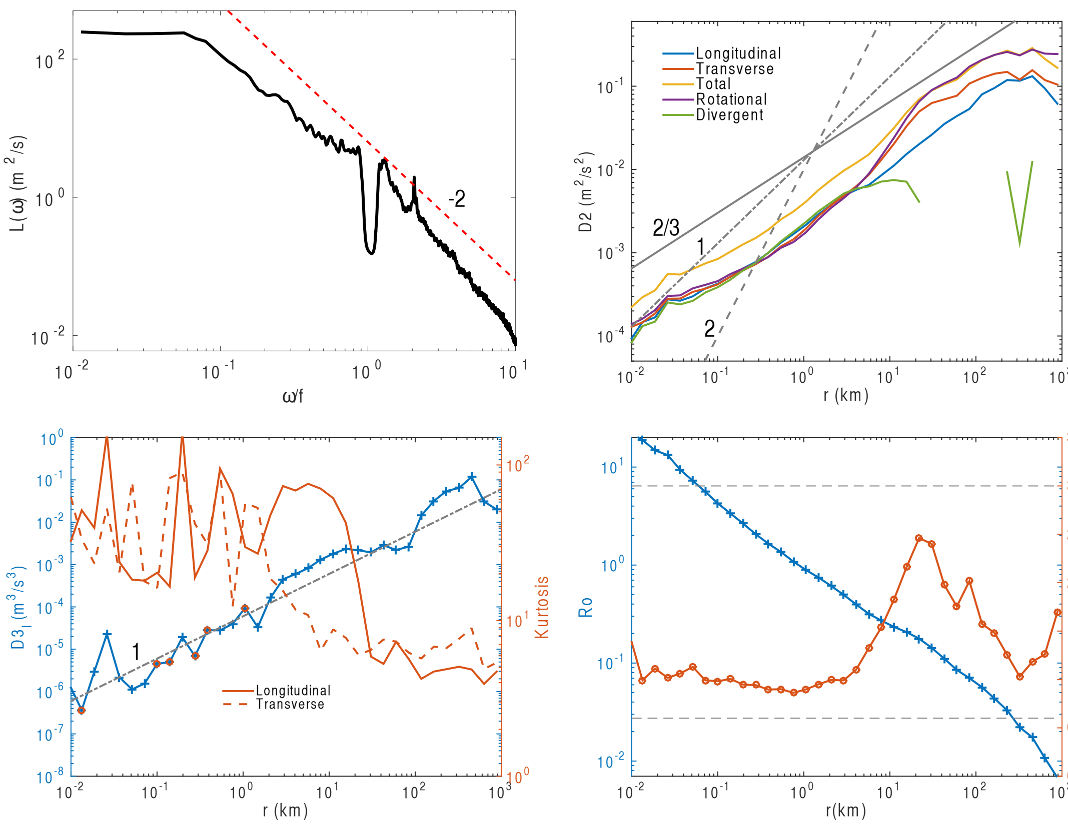

Figure S6: **Bandpass filtered results** (a), (b), (c) and (d) are the same as for Figure S5.

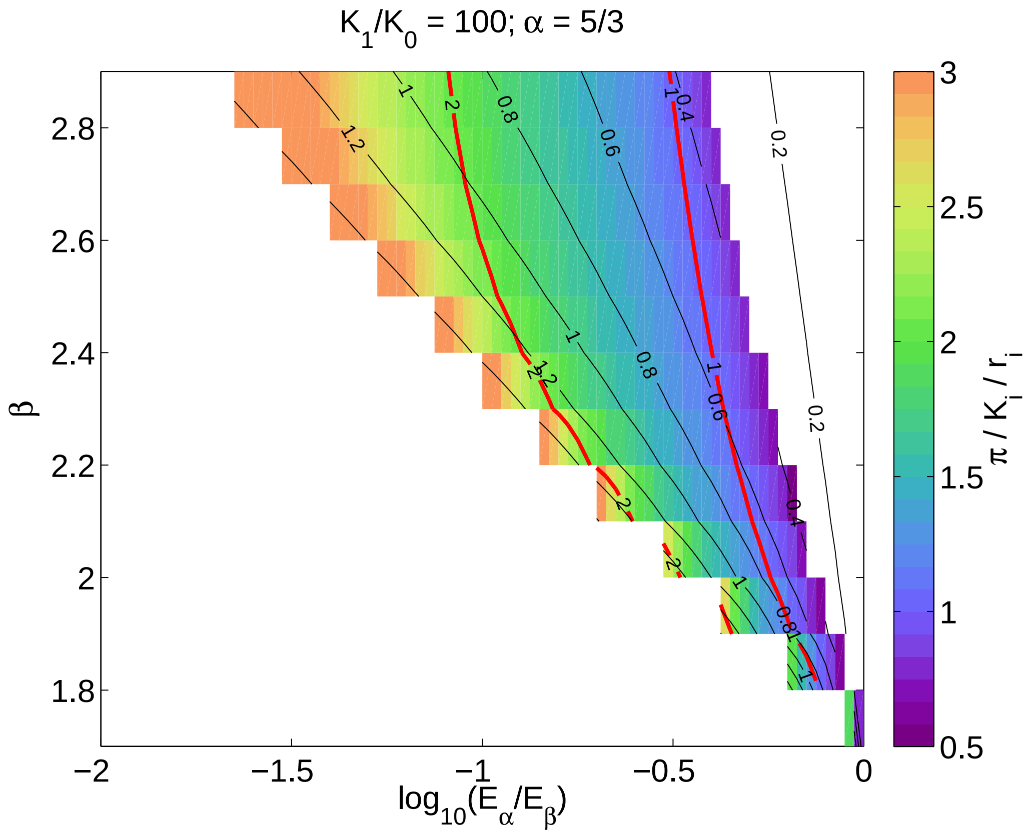

Figure S7: Colored contours show the ratio of the intersection length scale for the energy spectra to the intersection length scale ($\pi/(k_i r_i)$) for the structure functions as a function of slope and ratio of energy density at the the lower wavenumber cut off. The red colored contour lines represent the ratios of of 1 and 2. α was fixed to be 5/3 and β varies from 5/3 to 3. The marked black contour lines show the ratio of k_o to the intersection wavenumber (k_o/k_i) on a log scale.

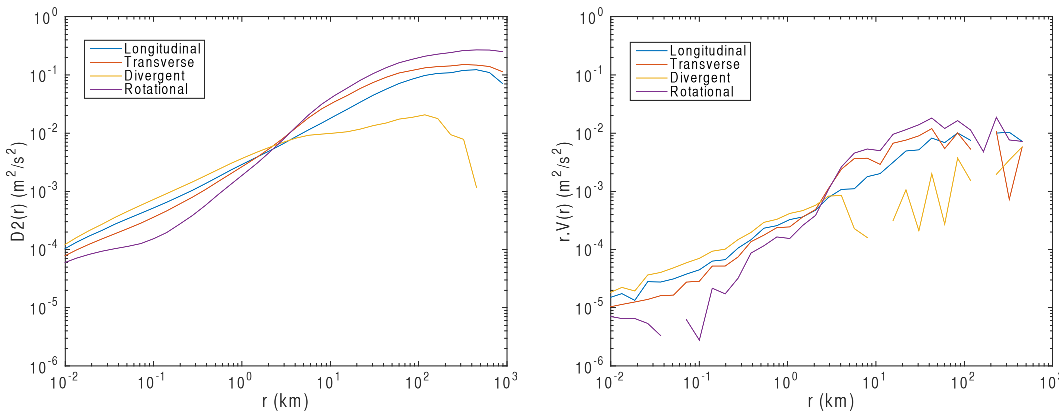

Figure S8: (Left) Smoothed second order velocity structure functions calculated from the original structure functions (Figure 2, main text). (Right) Signature functions corresponding to the smoothed second order velocity structure functions.